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Stability and vibrations of catenoid-shaped smectic films

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Abstract. Catenoid-shaped smectic films are spanned between two coaxial circular frames separated by a distance H. It is shown that there exists a critical height H^* such that below it two shapes of the catenoid are possible. The stability of these two shapes is analysed in terms of their vibrations. The spectrum of eigenfrequencies is calculated as a function of the catenoid height. It is shown that the frequency of the fundamental mode is real for the stable shape and imaginary for the other shape. Experimental study of vibrational eigenmodes performed on stable SCE4 films confirms this theoretical prediction.

PACS. 61.30.-v Liquid crystals – 62.30.+d Mechanical and elastic waves; vibrations – 03.40.-t Classical mechanics of continuous media: general mathematical aspects

1 Introduction

When molecular layers forming a free-standing smectic film are liquid (in two dimensions), the tension τ of the film is uniform and isotropic. The thickness of the film can be made uniform too and set in a wide range by an appropriate manipulation during its drawing. Such films, of uniform tension and thickness, behave as perfect membranes [1].

When smectic films are *flat*, their vibrations in vacuum obey the Helmholtz wave equation [2]. For this reason, smectic films have been used recently as a model system for an experimental verification of the theoretical predictions concerning the relationship between the shapes of flat drums and the spectra of their eigenfrequencies. In particular, it has been shown experimentally that two drums of different shapes invented by Gordon, Webb and Wolpert are isospectral [3].

The case of *curved* smectic films is much more complex for several reasons. First of all, for a given geometry of the frame, the equilibrium shapes of the film depend on the pressure difference Δp between its two sides. In the simplest case, when $\Delta p = 0$, the film must take the shape of a surface having everywhere zero mean curvature. This leads to the second difficulty because on a given frame more than one of such surfaces can be spanned so that one has to deal with the problem of the stability of possible shapes. The example of films spanned on two identical circular coaxial frames of radius R separated by a distance H (Fig. 1) is generic. On such a frame, smectic films can either take the shape of two flat discs spanned separately *on* the circular frames or the shape of a catenoid [4] – the unique minimal surface of revolution – spanned *between* the two frames. It has been shown in past [5] and in the accompanying paper [6] that for certain distances H, two different catenoids can be spanned between the frames. This is shortly reminded in Section 2.1 where we calculate possible shapes of the catenoid as a function of the distance H.

The principal aim of the present paper is to consider the problem of stability of these shapes with respect to

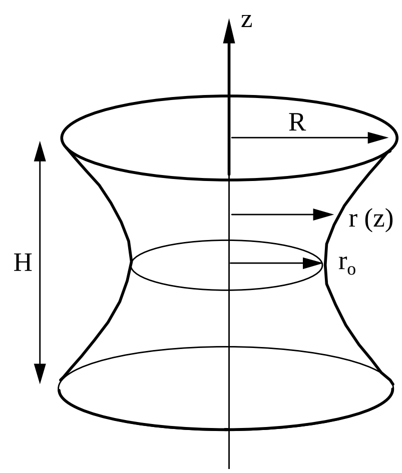

Fig. 1. Parameters of a surface of revolution.

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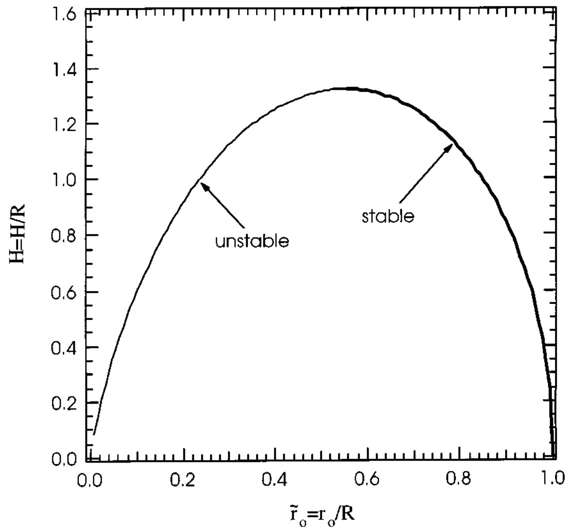

Fig. 2. Height of the catenoid *versus* its minimal radius. The radius R of the frame is used as the unit of length.

infinitesimal transverse deformations. This is done in Section 2.2 where, using the equation of motion proposed by Ben Amar and Patricio da Silva [7], we calculate the spectrum of vibrational eigenmodes. These calculations are made under assumption that smectic films are liquid in two dimensions so that only the capillary restoring force is taken into account. Nevertheless, it is important to note that smectic films can have an in-plane elasticity (SmB-like films) that can also contribute to the restoring force [8]. The theoretical analysis of Section 2 is completed in Section 3 by presentation of measurements of eigenfrequencies on SmC*-like films.

2 Theory of stability and vibrations of the catenoid

2.1 Possible shapes of the catenoid

Let $r = r(z)$ be the radius of the surface of revolution given as a function of the height z (Fig. 1). For the catenoid one has [4]:

$$r(z) = r_0 \mathrm{cosh}\left(\frac{z}{r_0}\right) \quad (1)$$

where r_0 corresponds to the minimal radius at $z = 0$. For a given radius R of the frames, this minimal radius depends on the distance H through the boundary condition:

$$\frac{H}{2} = r_0 \mathrm{arcch}\left(\frac{R}{r_0}\right) \cdot \quad (2)$$

Using the radius R as the unit of length and reduced variables $\tilde{r}_0 = r_0/R$ and $\tilde{H} = H/R$, one gets

$$\tilde{H} = 2\tilde{r}_0 \mathrm{arcch}\left(\frac{1}{\tilde{r}_0}\right) \cdot \quad (3)$$

The plot $\tilde{H}(\tilde{r}_0)$ is shown in Figure 2. It has a maximum $\tilde{H}^* = 1.3254$ for $\tilde{r}_0^* = 0.5524$. It means that for the distance H larger then $1.3254R$ no catenoid can be spanned between the two circular frames. For $\tilde{H} < \tilde{H}^*$, two catenoids of different radii can satisfy the boundary condition (3). Of course, at fixed H and R values, the catenoid with the larger r_0 value has smaller area, so it is energetically favoured.

It has been shown experimentally (see the previous paper [6]) that the right branch ($\tilde{r}_0^* < \tilde{r}_0 < 1$) of the plot $\tilde{H}(\tilde{r}_0)$ corresponds to stable shapes of the catenoid while the left branch ($0 < \tilde{r}_0 < \tilde{r}_0^*$) corresponds to unstable shapes. In experiments in which the smectic film is spanned between two circular frames, only the right branch of the plot $\tilde{H}(\tilde{r}_0)$ can be realised. When the distance H is increased above the critical value H^*, the catenoid collapses. As a result, one gets two flat films spanned separately on the two frames and a small smectic bubble is created in the middle of the gap between the frames.

It has been shown in reference [6] that in spite of its instability the left branch of the plot in Figure 2 can also be realised experimentally. For this purpose, one of the circular frames has to be replaced by a flat plate and the distance $H/2$ between the frame and the plate must be regulated in an appropriate way preventing the instability.

The analysis of vibrations presented in the next section will confirm the stable and unstable characters of the two possible shapes of the catenoid.

2.2 Vibrations of the catenoid

Let us consider a catenoid given by equation (1). Its vibrations involve a displacement

$$w(z, \phi, t) = e^{i\omega t} w(z, \phi) \quad (4)$$

in the direction normal to the surface. Due to this displacement, a surface element of the catenoid is submitted to two forces: the force due to the tension τ of the surface and the force of inertia due to its mass ρ per unit area. The equation of motion resulting from the balance of these forces has been written in [7] as:

$$\tau \frac{1}{r_0^2 \mathrm{ch}^2\theta}\left(\frac{\partial^2 w(\theta,\phi)}{\partial\theta^2} + \frac{\partial^2 w(\theta,\phi)}{\partial\phi^2}\right) + \tau\frac{2w(\theta,\phi)}{r_0^2 \mathrm{ch}^4\theta} + \rho\omega^2 w(\theta,\phi) = 0 \quad (5)$$

where $\theta = z/r_0$. The first term in this equation represents the Laplace force due to the mean curvature created by a non uniform displacement field $w(\theta, \phi)$. The second term represents the Laplace force due to the mean curvature created by an uniform displacement of the surface. The third term represents obviously the inertial force. Using

$$\omega_u = \frac{1}{r_0}\sqrt{\frac{\tau}{\rho}} \quad (6)$$

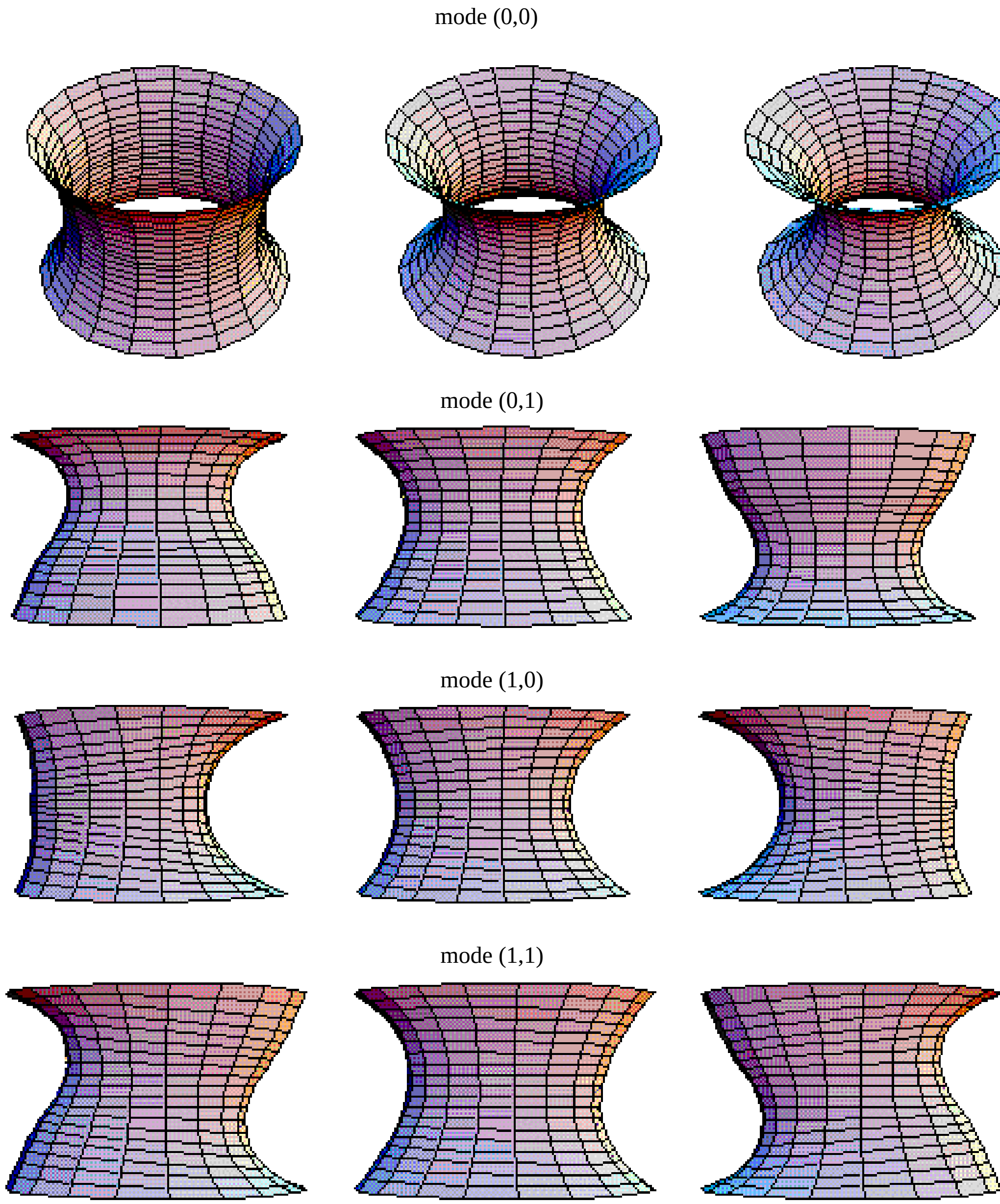

Fig. 3. Modes of vibrations of the catenoid.

as a unit of frequency, the equation (5) can be written as

$$\left(\frac{\partial^2 w(\theta,\phi)}{\partial\theta^2}+\frac{\partial^2 w(\theta,\phi)}{\partial\phi^2}\right)+\frac{2w(\theta,\phi)}{\mathrm{ch}^2\theta} + \mathrm{ch}^2\theta\tilde{\omega}^2 w(\theta,\phi)=0 \quad (7)$$

where

$$\tilde{\omega}=\frac{\omega}{\omega_u} \quad (8)$$

is a dimensionless frequency.

Due to the symmetry of revolution of the catenoid, the eigenmodes of the catenoid must be periodic with respect to ϕ and can be represented as:

$$\begin{aligned} w(\theta,\phi) &= w(\theta)\cos(n\phi) \quad (n=0,1,2,...),\\ w(\theta,\phi) &= w(\theta)\sin(n\phi) \quad (n=1,2,3,...), \end{aligned} \quad (9)$$

so that the equation (7) reduces to

$$\frac{\partial^2 w(\theta)}{\partial\theta^2}+\left(\frac{2}{\mathrm{ch}^2\theta}-n^2+\tilde{\omega}^2\mathrm{ch}^2\theta\right)w(\theta)=0. \quad (10)$$

The eigenvalues $\tilde{\omega}$ have been found by a numerical integration of this equation in intervals $(-\Theta/2,\Theta/2)$ with Θ varying from 0 to ∞. For a given Θ, the integration starts from the lower limit of the interval corresponding

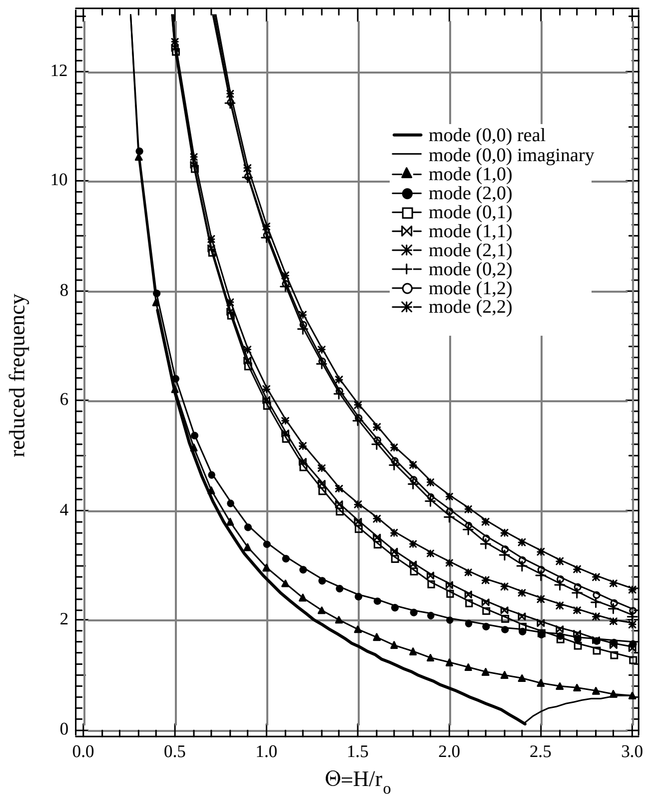

Fig. 4. Calculated frequency of eigenmodes.

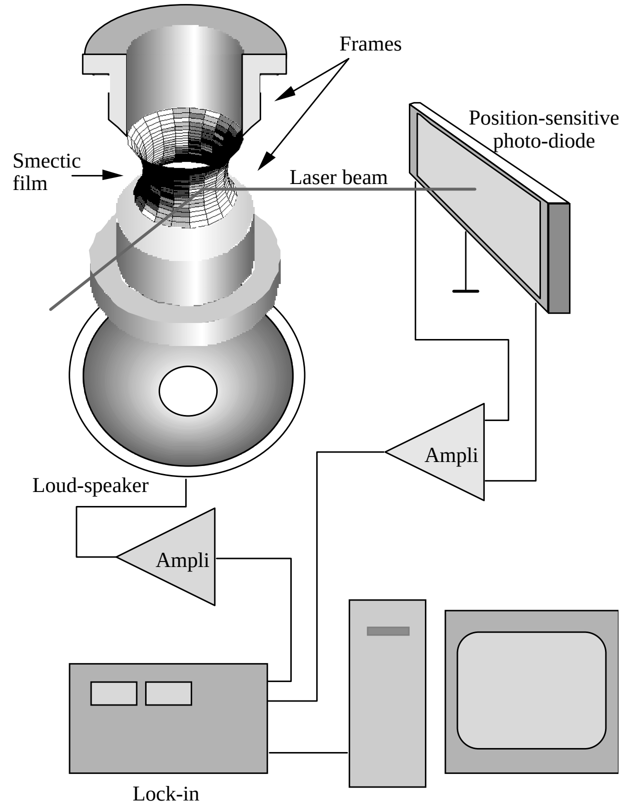

Fig. 5. Experimental set-up.

to the lower frame of the catenoid where obviously one has $w(-\Theta/2) = 0$. The choice of the first derivative $w'(-\Theta/2)$ is arbitrary because only the amplitude of the eigenfunction is affected by it. The parameter $\tilde{\omega}$ is then adjusted in a way that on the upper frame one has $w(\Theta/2) = 0$. By this means, for a given n, one finds a set of eigenvalues $\tilde{\omega}_{nm}$ corresponding to eigenfunctions $w_{nm}(\theta, \phi)$ with m nodes in the interval $(-\Theta/2, \Theta/2)$.

In Figure 3 we show a few eigenfunctions $w_{nm}(\theta)$ calculated by this procedure. The plot of eigenvalues $\tilde{\omega}_{nm}$ *versus* Θ is shown in Figure 4. Its most striking feature is the change of the sign of the eigenvalue $\tilde{\omega}_{00}^2$ at $\Theta^* \approx 2.4$. This means that for $\Theta < \Theta^*$, the frequency $\tilde{\omega}_{00}$ of the fundamental mode is real so that it has an oscillatory behaviour. For $\Theta > \Theta^*$ the frequency $\tilde{\omega}_{00}$ is imaginary what corresponds to an exponential divergence of the mode – the catenoid becomes unstable. All other modes have no singularity at Θ^*. This instability can be shown by an explicit resolution of equation (10) at zero $\tilde{\omega}$ and for $n = 0$:

$$w_{00} = A(\theta \text{th } \theta - 1). \tag{11}$$

The boundary conditions: $w(\pm\Theta/2) = 0$ gives $\Theta^* \text{th } \Theta^* = 1$, which is exactly the condition for a double root of equation (3): both r_0 values coincide. This gives precisely $\Theta^* = 2.398$.

In order to understand this plot in terms of experimental parameters let us remind that $\theta = z/r_0$ so that $\Theta = H/r_0$. On the other hand, H depends on r_0 (Eq. (2) and Fig. 2). When r_0 varies between R and $r_0^* = \tilde{r}_0^* R$, H varies from 0 to $H = \tilde{H}^* R$ (the right branch of the plot corresponding to the stable form of the catenoid). In these conditions, the parameter Θ varies from 0 to $\tilde{H}^*/\tilde{r}_0^* = 2.4$. This ratio corresponds exactly to the value Θ^* defining the interval $(0, \Theta^*)$ in which the frequency $\tilde{\omega}_{00}$ of the fundamental mode is real (Fig. 4). Now, when the minimal radius of the catenoid r_0 varies between $r_0^* = \tilde{r}_0^* R$ and 0, H varies from $H^* = \tilde{H}^* R$ to 0 on the unstable branch of the plot in Figure 2. The resulting variation of the parameter Θ in Figure 4 is between Θ^* and ∞. The frequency $\tilde{\omega}_{00}$ is now imaginary as expected.

Let us note at the end of this section that plots of higher modes with n or/and m different from zero have no singularity whatever value of Θ. This means that the catenoid is only unstable with respect to the fundamental mode. This analysis proves that the stability of the catenoid derived from a variational argument of minimisation of energy is consistent with a dynamical approach of the vibration spectrum.

3 Experiments on vibrations of the catenoid

The set-up used for the production of catenoid-shaped smectic films, excitation and detection of vibrations is depicted in Figure 5. Two cylindrical parts with sharp rims of radius $R = 5$ mm form circular frames of the catenoid. The distance H between them is controlled with the accuracy of 0.02 mm by means of a translation stage. In order

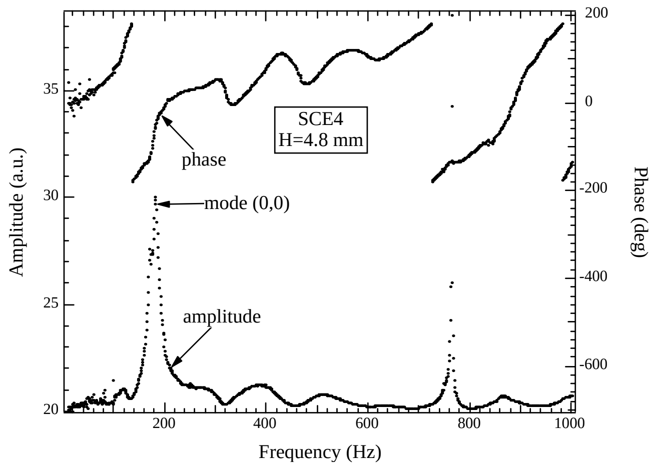

Fig. 6. Typical frequency scan.

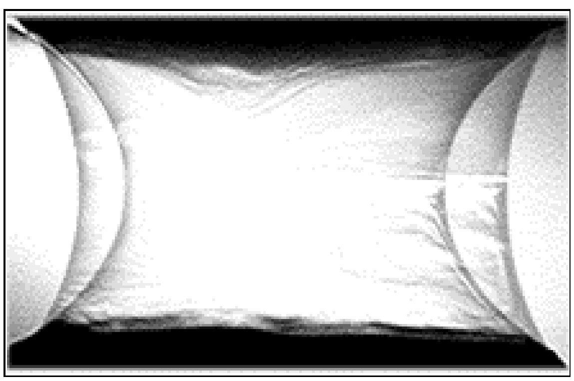

Fig. 7. Stroboscopic picture of the fundamental mode.

to draw a film, the frames are wet with a small amount of liquid crystal (SCE4 from BDH) [9] and brought into contact. Then, they are slowly pulled apart; the catenoid-shaped film forms between them. The typical thickness of films estimated from its optical aspect was between 10 and 30 layers. The knowledge of exact thickness is not necessary because experiments are performed at ambient pressure so that the inertia of the air put into motion is much larger than the inertia of the film itself. Vibrations of the film are excited acoustically by a loud-speaker situated below frames. The detection is optical. A laser beam reflected obliquely by the film is sent on a position-sensitive photo-diode. The amplitude and the phase of the signal from the photodiode are detected by a lock-in amplifier which also produces the ac voltage used for the excitation.

The typical frequency scan is represented in Figure 6. It shows the presence of a well defined peak centred at 183 Hz. Stroboscopic movie-pictures taken with a TV hand-camera allow to determine the shape of this mode and its indices (n, m). The mode shown on the photograph in Figure 7 has no nodes in z-direction and can be identified as the fundamental mode (0, 0). The frequency of this mode was measured as a function of the height H.

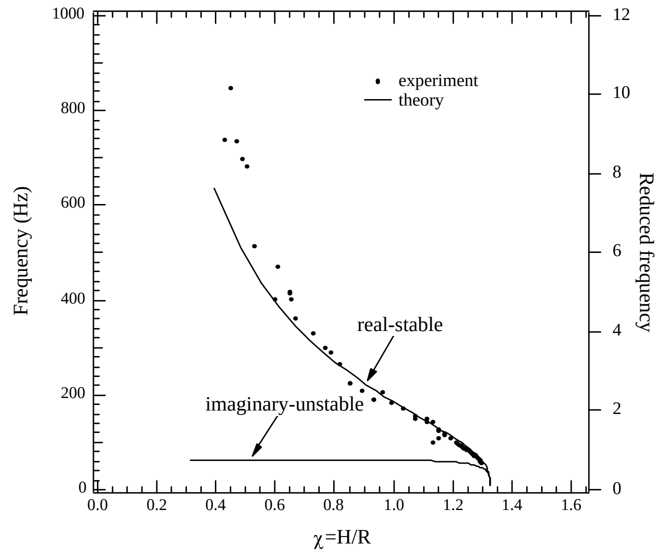

Fig. 8. Frequency of the fundamental mode *versus* the reduced height of the catenoide.

Results are plotted in Figure 8 (using the left axis for the frequency) *versus* the reduced height $\tilde{H} = H/R$. This experimental plot has to be compared with calculations from the previous section. For this purpose, the plot from Figure 4. can not be used such as it is because the Θ axis does not correspond directly to the reduced height $\tilde{H}$. Therefore, the eigenvalues $\tilde{\omega}_{00}$ have been recalculated as a function of $\tilde{H}$. In practice, one sets the value of r_0 between $\tilde{r}_0^* R$ and R and calculates H using equation (2). Knowing H, R and r_0, one gets values Θ and $\tilde{H}$. The integration procedure gives now the dimensionless eigenvalues $\tilde{\omega}_{00}$ as a function of the reduced height $\tilde{H}$. They are plotted in Figure 8 using the right axis for which the range of reduced frequencies was adjusted in a way to obtain the best agreement with experimental points.

4 Discussion and conclusions

It is important to note that this adjustment was made in a way to get the best agreement in the vicinity of the critical height $\tilde{H}^*$. One can say that the value of the critical height found experimentally is in good agreement with the one found by calculations of the eigenfrequencies. However, for $\tilde{H} < 0.8$, the difference between the experimental and theoretical frequencies seems to be growing. The most plausible explanation of this discrepancy is based on the fact that experiments have been performed at ambient pressure so that the inertia of air is much larger than the one of the film itself. In these conditions, the equation of motion (5) can only be considered as an approximation in which the parameter ρ has to be considered as an effective density taking into account the mass of the air accompanying the film in its motion. When the height H of the catenoid decreases, this effective mass decreases too so that the frequency of eigenmodes increases. In order to avoid this difficulty, experiments could be made in vacuum. In such a case, instead of the acoustical excitation, electrostatic forces produced by a pin-point electrode could be used. The advantage of this type of excitation is that by choosing an appropriate position of the electrode all kinds of modes (m, n) could be excited.

Measurements of eigenmodes presented here were made using the stable shape of the catenoid. However, as pointed out in the previous paper, the unstable shape can be realised too when one of the circular frames is replaced by a flat plate. Even if in these "half-catenoid" geometry only the modes (n, m) with n odd can be excited, it could be interesting to extend the measurements of eigenfrequencies beyond the stability limit Θ^*.

We hope that this work opens the way to the verification of the isospectrality of minimal surfaces proposed in [7].

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9. For the purpose of the present experiments the SCE4 liquid crystal showed experimentally better features than 8CB in terms of the facility in drawings films between the two circular frames. This choice of the liquid crystal having the SmC* phase at room temperature opens the question about the possible coupling between the vibrations of the film and the vector field **c** of the molecular tilt. The surface energy density of the distorsion of the field **c** can be estimated as Kdq^2 where K is the elastic constant of the order of 10^{-7} dyn and d is about 10^{-5} cm. It has to be compared with the film tension of the order of 50 dyn/cm. As conclusion, these energies are of the same order of magnitude when $q^2 = 50 \times 10^{12}$ cm^{-2}. It means that the wavelength of the **c** field winding should be 10^{-6} cm. Such a distorsion never occurs in practice so that this coupling can be neglected.